



TITLE:

THE ACOUSTIC WAVE PROPAGATION IN TWO UNBOUNDED MEDIA

AUTHOR(S):

KADOWAKI, MITSUTERU

CITATION:

KADOWAKI, MITSUTERU. THE ACOUSTIC WAVE PROPAGATION IN TWO UNBOUNDED MEDIA. 数理解析研究所講究録 1995, 894: 29-42

ISSUE DATE:

1995-01

URL:

<http://hdl.handle.net/2433/84439>

RIGHT:

THE ACOUSTIC WAVE PROPAGATION IN TWO UNBOUNDED MEDIA

MITSUTERU KADOWAKI

門脇 光輝 (筑波大学・大学院)

1. Introduction.

In the present paper we study the limiting absorption and amplitude principle for the acoustic wave operators in two unbounded media. We assume that the propagation speed is discontinuous at the interface and the equilibrium density is 1.

Let $n \geq 2$ and $x = (y, z) \in \mathbf{R}^{n-1} \times \mathbf{R}$. The following equation describes the wave propagation here :

$$(1.1) \quad \partial_t^2 u(t, x) - a(x)^2 \Delta u(t, x) = 0, \quad (t, x) \in \mathbf{R} \times \mathbf{R}^n,$$

where $a(x)$ is a propagation speed. When we consider the limiting amplitude principle, we deal with the asymptotic behaviour (as $t \rightarrow +\infty$) of the solutions of the following Cauchy problem

$$(1.2) \quad \begin{cases} \partial_t^2 u(t, x) - a(x)^2 \Delta u(t, x) = \exp(-it\sqrt{\omega}) f(x) & (t, x) \in \mathbf{R}_+ \times \mathbf{R}^n, \\ u(0, x) = \partial_t u(0, x) = 0, \end{cases}$$

where $\omega > 0$. But we can not deal with $n = 2$ (see (3.13)).

We make the assumptions for the interface separating two media and $a(x)$.

Let $\varphi_0(y) = b|y|$ and $\varphi(y) \in C^1(\mathbf{R}^{n-1} \setminus \{0\})$, where $b \geq 0$. We assume that $\varphi(y)$ describes the interface and satisfies

$$(A.0) \quad \sum_{|\alpha| \leq 1} |y|^{|\alpha|} |\partial^\alpha (\varphi(y) - \varphi_0(y))| = O(|y|^{-\theta}) \quad (|y| \rightarrow \infty),$$

for some $\theta > 0$, and

$$(A.1) \quad \sum_{|\alpha| \leq 1} |y|^{|\alpha|} |\partial^\alpha \varphi(y)| = O(|y|^{-\sigma}) \quad (|y| \rightarrow 0).$$

where $0 < \sigma < 1/2$. For $\varphi(y)$, we use the following notation :

$$\begin{aligned} \Omega_+ &= \{x = (y, z) : z > \varphi(y)\}, \\ \Omega_- &= \{x = (y, z) : z < \varphi(y)\}, \\ S &= \{x = (y, z) : z = \varphi(y)\}. \end{aligned}$$

We denote the unit normal vector at the point $x \in S$ by $\nu = (\nu_1, \nu_2, \dots, \nu_z)$ with $\nu_z > 0$.

The propagation speed $a(x) > 0$ is assumed to satisfy the following : for some $c > 1$,

$$(A.2) \quad 1/c < a(x) < c$$

and there exist $a_{\pm} > 0$, $a_L^{\pm}(x) \in \mathbf{B}^1(\Omega_{\pm})$ and $a_S(x) \in L^{\infty}(\mathbf{R}^n)$ such that $a(x)$ is decomposed as

$$(A.3) \quad \begin{cases} a(x) &= a_{\pm} + a_L^{\pm}(x) + a_S(x) \quad (x \in \Omega_{\pm}), \\ \sum_{|\alpha| \leq 1} |x|^{|\alpha|} |\partial^{\alpha} a_L^{\pm}(x)| &= O(|x|^{-\theta}) \quad (|x| \rightarrow \infty, x \in \Omega_{\pm}), \\ a_S(x) &= O(|x|^{-\theta-1}) \quad (|x| \rightarrow \infty) \end{cases}$$

for some $\theta > 0$.

Under (A.0) ~ (A.3), we show the nonexistence of eigenvalues and the limiting absorption and amplitude principle of the acoustic operator $-a(x)^2 \Delta$ for (1.1).

There are many works dealing with the acoustic wave propagation problem with the discontinuous propagation speed at the interface separating media. Eidus [6] proved the limiting absorption and amplitude principle for two unbounded media problem with the interface satisfying the following conditions : for any $x \in S$

$$(1.3) \quad \nu_z \geq C_1 > 0,$$

$$(1.4) \quad |x \cdot \nu| \leq C_2,$$

where $C_j > 0$, ($j = 1, 2$), are independent of $x \in S$. For example,

$$\varphi(y) \in C^1(\mathbf{R}^{n-1}), \quad \varphi(y) = \frac{\sin |y|}{|y|} \quad (|y| > 1), \quad \varphi_0(y) = 0$$

satisfies (1.3) and (1.4), but not satisfies (A.0). We can also deal with the following interface not satisfying (1.3),

$$\varphi(y) = |y|^{-\sigma}, \quad \varphi_0(y) = 0$$

where, $0 < \sigma < 1/2$.

The propagation speed considered in Eidus [6] is a piecewise constant function while we can perturb the propagation speed. Wilcox [18] considered two stratified fluids in a half space and established the eigenfunction expansion theorem. Ben-Artzi [1], Weder [15]~[17] and Dermenjian and Guillot [3] considered perturbed stratified fluids problems. They showed the limiting absorption principle by the approach of Wilcox [18]. Kikuchi and Tamura [9] also proved the limiting amplitude principle for perturbed stratified fluids. On the other hand there are some works dealing with

the case where the equilibrium density is discontinuous at the interface separating media, for example, Debièvre and Pravica [2] and Wilcox [18].

In order to show the limiting absorption and amplitude principle for our operator, we use Mourre's method. This method was first developed by Mourre [10] to prove the limiting absorption principle for 3-body Schrödinger operators (see also Perry, Sigal and Simon [11] and Tamura [13]). In Froese and Herbst [8], they showed by Mourre's method that N-body Schrödinger operators have no positive eigenvalues. Iwashita [8] and Weder [14] showed the limiting absorption principle for first order symmetric systems. For the acoustic wave operators in perturbed stratified fluids, Debièvre and Pravica [2] obtained the similar results as in [8] and [14]. Tamura [12] used Mourre's method in order to prove the limiting amplitude principle for the acoustic wave operators (see also Kikuchi and Tamura [9]).

We now define the acoustic operator L as

$$(1.5) \quad L = -a(x)^2 \Delta$$

Under the above assumptions, (A.0)~(A.3), L is a symmetric operator in the Hilbert space $L^2(\mathbf{R}^n; a^{-2}(x)dx)$ and admits a unique self-adjoint realization. We denote by the same notation L this self-adjoint realization. Then L is a positive operator (zero is not an eigenvalue) and the domain $D(L)$ is given by $D(L) = H^2(\mathbf{R}_x^n), H^s(\mathbf{R}_x^n)$ being the Sobolev space of order s over \mathbf{R}_x^n . We also denote by $R(z; L)$ the resolvent $(L - z)^{-1}$ of L for $\text{Im} z \neq 0$.

We need several notations to describe our results. Let L^2 be the usual L^2 space defined on \mathbf{R}^n , with the inner product

$$\langle u, v \rangle = \int_{\mathbf{R}^n} u(x) \overline{v(x)} dx$$

and the corresponding norm $|\cdot|_0$. For $\alpha \in \mathbf{R}$ let L_α^2 be the weighted L^2 space defined by

$$L_\alpha^2 = \{u(x) : \langle x \rangle^\alpha u(x) \in L^2(\mathbf{R}_x^n)\}, \langle x \rangle = (1 + |x|^2)^{1/2},$$

with the norm

$$|u|_\alpha^2 = \int_{\mathbf{R}^n} \langle x \rangle^{2\alpha} |u(x)|^2 dx.$$

Let $A : L_\alpha^2 \rightarrow L_\beta^2$ be a bounded operator. We denote by $\|A\|_{\alpha \rightarrow \beta}$ the operator when norm considered as an operator from L_α^2 to L_β^2 . If, in particular, $A : L^2 \rightarrow L^2$ is considered as an operator from L^2 into itself, then its norm is denoted by the simplified notation $\|A\|$.

The main results are

THEOREM 1.1. *Assume that (A.0) ~ (A.3). Then*

(i) *L has no eigenvalues.*

(ii) Let $\lambda_0 > 0$ and $\alpha > 1/2$. Then for any compact interval $I \subset \mathbf{R}_+$ containing λ_0 , there exists a positive constant $C = C(I, \alpha)$ such that

$$\| \langle x \rangle^{-\alpha} R(\lambda \pm i\kappa; L) \langle x \rangle^{-\alpha} \| \leq C,$$

for $\lambda \in I$, $0 < \kappa < 1$.

(iii) For every $\lambda > 0$ and $\alpha > 1/2$, following two limits

$$R(\lambda \pm i0; L) = \lim_{\kappa \downarrow 0} R(\lambda \pm i\kappa; L),$$

exist in the uniform operator topology of $\mathfrak{B}(L_\alpha^2, L_{-\alpha}^2)$. Moreover $R(\lambda \pm i0; L)$ are locally Hölder continuous.

THEOREM 1.2. Assume that (A.0) \sim (A.3) and $n \geq 3$. Let $\alpha > 1$ and $\beta > 1/2$. Then there exists d , $0 < d < 1/2$, such that

$$\|R(\lambda \pm i0; L)\|_{\beta \rightarrow -\alpha} = O(\lambda^{-d}), \quad (\lambda \rightarrow 0).$$

By Theorem 1.1 and 1.2, we have the following theorem (see Eidus [5] or Tamura [8]).

THEOREM 1.3 (LIMITING AMPLITUDE PRINCIPLE). Assume that (A.0) \sim (A.3) and $n \geq 3$. Let $u = u(t, x)$ be the solution of (1.2) with $f \in L_\beta^2$, $\beta > 1/2$. Then $u(t, x)$ behaves like

$$u = \exp(-it\sqrt{\omega}) R(\omega + i0; L) f + o(1), \quad (t \rightarrow \infty)$$

strongly in $L_{-\alpha}^2$, $\alpha > 1$.

2. The limiting absorption principle.

We consider only the case $1 = a_-^{-2} < a_+^{-2}$. The other cases can be proved similarly. We define the self-adjoint operator $H(\lambda)$ on L^2 by

$$\begin{cases} H(\lambda) = -\Delta - \lambda(a^{-2}(x) - 1) \\ D(H(\lambda)) = H^2(\mathbf{R}^n). \end{cases}$$

Then we have

$$R(\lambda \pm i\kappa; L) = Q(\lambda, \pm i\kappa; H(\lambda)) a^{-2}(x),$$

where $Q(\lambda, \pm i\kappa; H(\lambda)) = (H(\lambda) - \lambda \mp i\kappa a^{-2}(x))^{-1}$.

We apply Mourre's commutator method to $H(\lambda)$ on L^2 (see Tamura [12] or Kikuchi and Tamura [9]).

By (A.3), we can decompose $a^{-2}(x) = E_L^\pm(x) + E_S(x)$ ($x \in \Omega_\pm$) in such a way that

$$\sum_{|\alpha| \leq 1} |x|^{|\alpha|} |\partial^\alpha (E_L^\pm(x) - a_\pm^{-2})| = O(|x|^{-\theta}), \quad (|x| \rightarrow \infty, x \in \Omega_\pm),$$

$$E_S(x) = O(|x|^{-1-\theta}), \quad (|x| \rightarrow \infty).$$

Let A be the generator of the dilation unitary group;

$$A = \frac{1}{2i}(x \cdot \nabla + \nabla \cdot x).$$

We define the commutator $i[H(\lambda), A]$ as a form on $H^2(\mathbf{R}^n) \cap D(A)$ as follows; For $u, v \in H^2(\mathbf{R}^n) \cap D(A)$

$$\begin{aligned} & \langle i[H(\lambda), A]u, v \rangle \\ &= i(\langle Au, H(\lambda)v \rangle - \langle H(\lambda)u, Av \rangle). \end{aligned}$$

Our main calculation is the integration by parts. Then, an integration on S appears because of the discontinuity of $a(x)$. In order to estimate this integration, we use the following lemma

LEMMA 2.1. Let $s > 1/2$. For $u \in \mathbf{S}(\mathbf{R}^n)$ (Schwartz space), we define

$$(T_\varphi u)(y) = u(y, \varphi(y)).$$

T_φ has an extension to a bounded operator from $H^s(\mathbf{R}^n)$ to $L^2(\mathbf{R}^{n-1})$.

PROOF: we show that :

$$(2.1) \quad |T_\varphi u|_{L^2(\mathbf{R}^{n-1})} \leq C|u|_{H^s(\mathbf{R}^n)}$$

for $u \in \mathbf{S}(\mathbf{R}^n)$. Let $u \in \mathbf{S}(\mathbf{R}^n)$. Then $u(y, \varphi(y))$ is represented as

$$(2.2) \quad u(y, \varphi(y)) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} e^{i\varphi(y) \cdot \zeta} (\mathfrak{F}_z u)(y, \zeta) d\zeta,$$

where $\xi = (\eta, \zeta) \in \mathbf{R}_\eta^{n-1} \times \mathbf{R}_\zeta = \mathbf{R}_\xi^n$ and \mathfrak{F}_z is the Fourier transform in \mathbf{R}_z . By Schwartz's inequality, we have

$$(2.3) \quad \begin{aligned} & |(T_\varphi u)(y)|^2 \\ & \leq (2\pi)^{-1} \int_{-\infty}^{+\infty} (1 + \zeta^2)^{-s} d\zeta \int_{-\infty}^{+\infty} |(1 + \zeta^2)^{s/2} (\mathfrak{F}_z u)(y, \zeta)|^2 d\zeta. \end{aligned}$$

By integrating both sides of the above inequality over \mathbf{R}^{n-1} , we obtain (2.1). ■

Moreover, a singularity at $y = 0$ for $\varphi(y)$ (see (A.2)) appears. We analyze this singularity by the argument using Hölder inequality and the following Sobolev's theorem

LEMMA 2.2 (SOBOLEV'S THEOREM). Suppose that

$$1/2 - l/m = 1/q, 2 < q < \infty.$$

Then we have the embedding

$$H^l(\mathbf{R}^m) \hookrightarrow L^q(\mathbf{R}^m).$$

The following lemmas play an important role in the proof of the theorem 1.1 (see Mourre [10] and Perry, Sigal and Simon [11]).

LEMMA 2.3. the form $i[H(\lambda), A]$ defined on $H^2(\mathbf{R}^n) \cap D(A)$ is extended to a bounded operator from $H^1(\mathbf{R}^n)$ to $H^{-1}(\mathbf{R}^n)$ which is denoted by $i[H(\lambda), A]^0$. Moreover we have

$$\begin{aligned} i[H(\lambda), A]^0 &= -2\Delta + \lambda((x \cdot \nabla E_L) - E_S x \cdot \nabla + \nabla^* \cdot E_S x - n E_S \\ &\quad - T_\varphi^*(E_L^{+0} - E_L^{-0})(y \cdot \nabla_y \varphi(y) - \varphi(y))T_\varphi), \end{aligned}$$

where $E_L^\pm = E_L(y, \varphi(y) \pm 0)$.

LEMMA 2.4. Let $\lambda_0 > 0$ and $0 < \delta < \min(1, \lambda_0/4)$ and take $f_\delta(p) \in C_0^\infty(\mathbf{R})$, $0 \leq f_\delta \leq 1$ such that f_δ has support in $(\lambda_0 - 3\delta, \lambda_0 + 3\delta)$ and $f_\delta = 1$ on $[\lambda_0 - 2\delta, \lambda_0 + 2\delta]$. Then, there exist a positive constant α and a compact operator K on L^2 which depend on only λ_0 such that

$$\begin{aligned} (2.4) \quad f_\delta(H(\lambda))i[H(\lambda), A]^0 f_\delta(H(\lambda)) \\ \geq \alpha f_\delta(H(\lambda))^2 + f_\delta(H(\lambda))K(\lambda)f_\delta(H(\lambda)) \end{aligned}$$

for $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$, in the form sense.

SKETCH OF THE PROOF OF LEMMA 2.3: By integration by parts, we have

$$\begin{aligned} (2.5) \quad &< i[H(\lambda), A]u, u > \\ &= 2 < \nabla u, \nabla u > + \lambda < (x \cdot \nabla E_L)u, u > \\ &\quad - \lambda < T_\varphi^*(\chi_{|y|<r}(y \cdot \nabla_y \varphi - \varphi)(E_L^{+0} - E_L^{-0}))T_\varphi u, u > \\ &\quad + < T_\varphi^*(\chi_{|y|>r}(y \cdot \nabla_y \varphi - \varphi)(E_L^{+0} - E_L^{-0}))T_\varphi u, u > \\ &\quad - n\lambda < E_S u, u > - \lambda < E_S u, x \cdot \nabla u > - \lambda < x \cdot \nabla u, E_S u >, \end{aligned}$$

for any $r > 0$.

We define an operator R_φ^r as

$$R_\varphi^r u = \chi_{|y|<r}(y)(y \cdot \nabla_y \varphi(y) - \varphi(y))^{1/2} T_\varphi u.$$

We show that R_φ^r is a bounded operator from $H^1(\mathbf{R}^n)$ to $L^2(\mathbf{R}^{n-1})$.

If $0 < \sigma < 1/2$, there exist some $s > 1/2$ and $p > n - 1$ such that

$$(2.6) \quad n - 1 - \sigma p > 0,$$

$$(2.7) \quad \frac{n-1}{2p} + s = 1.$$

By Hölder inequality, we have

$$\begin{aligned} (2.8) \quad &|R_\varphi^r u|_{L^2(\mathbf{R}^{n-1})}^2 \\ &\leq C \int_{-\infty}^{+\infty} (1 + \zeta^2)^s \int_{|y|<r} |y \cdot \nabla_y \varphi - \varphi| |(\mathfrak{F}_z u)(y, \zeta)|^2 dy d\zeta \\ &\leq C r^{\frac{n-1-\sigma p}{p}} \int_{-\infty}^{+\infty} (1 + \zeta^2)^s |(\mathfrak{F}_z u)(\cdot, \zeta)|_{L^{\frac{2p}{p-1}}(\mathbf{R}^{n-1})}^2 d\zeta, \end{aligned}$$

where $C > 0$ is independent of r .
 Lemma 2.2 implies that

$$(2.9) \quad |\mathfrak{F}_z u(\cdot, \zeta)|_{L^{\frac{2p}{p-1}}(\mathbf{R}^{n-1})} \leq C |\mathfrak{F}_z u(\cdot, \zeta)|_{H^{\frac{n-1}{2p}}(\mathbf{R}^{n-1})}.$$

It follows from (2.7), (2.8) and (2.9) that

$$(2.10) \quad |R_\varphi^r u|_{L^2(\mathbf{R}^{n-1})} \leq Cr^{\frac{n-1-\sigma p}{2p}} |u|_{H^1(\mathbf{R}^n)},$$

where C is independent of r .

It follows from Lemma 2.1 that T_φ^* is a bounded operator from $L^2(\mathbf{R}^{n-1})$ to $H^{-1}(\mathbf{R}^n)$. Moreover, since $E_S x \cdot \nabla$ is a bounded operator from $H^1(\mathbf{R}^n)$ to $L^2(\mathbf{R}^n)$, the adjoint operator $(\nabla)^* \cdot x E_S$ is a bounded operator from $L^2(\mathbf{R}^n)$ to $H^{-1}(\mathbf{R}^n)$.

Hence, from (2.5) and (2.10), the form $i[H(\lambda), A]$ has an extension to a bounded operator $i[H(\lambda), A]^0$ on from $H^1(\mathbf{R}^n)$ to $H^{-1}(\mathbf{R}^n)$. ■

SKETCH OF THE PROOF OF LEMMA 2.4: We simply write f_δ instead of $f_\delta(H(\lambda))$. By Lemma 2.3, we have for $u \in L^2$,

$$\begin{aligned} & \langle f_\delta i[H(\lambda), A]^0 f_\delta u, u \rangle \\ &= 2 \langle -\Delta f_\delta u, f_\delta u \rangle + \lambda \langle ((x \cdot \nabla E_L) - n E_S) f_\delta u, f_\delta u \rangle \\ &- \langle x \cdot \nabla f_\delta u, E_S f_\delta u \rangle - \langle E_S f_\delta u, x \cdot \nabla f_\delta u \rangle \\ &- \langle (T_\varphi)^* \chi_{|y|>r} (y \cdot \nabla_y \varphi - \varphi) (E_L^{+0} - E_L^{-0}) T_\varphi f_\delta u, f_\delta u \rangle \\ &- \langle (R_\varphi^r)^* (E_L^{+0} - E_L^{-0}) R_\varphi^r f_\delta u, f_\delta u \rangle. \end{aligned}$$

Let $0 < r \ll 1$. By (2.9), there exists a positive number C independent of r such that,

$$\begin{aligned} & |\lambda \langle (R_\varphi^r)^* (E_L^{+0} - E_L^{-0}) R_\varphi^r f_\delta u, f_\delta u \rangle| \\ &= |\lambda \langle (E_L^{+0} - E_L^{-0}) R_\varphi^r f_\delta u, R_\varphi^r f_\delta u \rangle_{L^2(\mathbf{R}^{n-1})}| \\ &\leq Cr^{\frac{n-1-\sigma p}{p}} (|\nabla f_\delta u|_0^2 + |f_\delta|_0^2). \end{aligned}$$

Let $E_0(x) = a_\pm^{-2}(x \in \Omega_\pm)$. Then we have

$$\begin{aligned} & 2 \langle -\Delta f_\delta u, f_\delta u \rangle - \lambda \langle (R_\varphi^r)^* (E_L^{+0} - E_L^{-0}) R_\varphi^r f_\delta u, f_\delta u \rangle \\ &\geq \left(\frac{(2 - Cr^{\frac{n-1-\sigma p}{p}}) \lambda_0}{4} - Cr^{\frac{n-1-\sigma p}{p}} \right) |f_\delta u|_0^2 \\ &+ (2 - Cr^{\frac{n-1-\sigma p}{p}}) \lambda \langle (E - E_0) f_\delta u, f_\delta u \rangle. \end{aligned}$$

Take $h \in C_0^\infty(\mathbf{R})$, $0 \leq h \leq 1$ such that $h = 1$ on $(\lambda_0/4, 7\lambda_0/4)$. Using h , we define an operator $K(\lambda)$ as

$$K(\lambda) = \sum_{j=1}^4 K_j(\lambda)$$

where,

$$\begin{aligned} K_1(\lambda) &= \lambda(h(H(\lambda))((x \cdot \nabla E_L) - nE_S + (2 - Cr^{\frac{n-1-\sigma p}{p}})(E - E_0))h(H(\lambda)), \\ K_2(\lambda) &= -\lambda h(H(\lambda))E_S x \cdot \nabla h(H(\lambda)), \\ K_3(\lambda) &= -\lambda h(H(\lambda))\nabla^* \cdot x E_S h(H(\lambda)), \\ K_4(\lambda) &= -\lambda h(H(\lambda))T_\varphi^*(E_L^{+0} - E_L^{-0})\chi_{|y|>r}(y \cdot \nabla \varphi - \varphi)T_\varphi h(H(\lambda)). \end{aligned}$$

For each $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$, $K_j(\lambda) (j = 1, 2, 3, 4)$ is a compact operator on L^2 . ■

Using Lemma 2.3 and Lemma 2.4, we can prove theorem 1.1(i) in the same way as in Froese and Herbst [7] and Appendix I of Tamura [12].

Next, we consider the proof of Theorem 1.1(ii) and (iii).

Following Tamura [12], we consider cut off functions, $\chi_n(x) \in C_0^\infty(\mathbf{R}^n)$ such that $\chi_n(x)$ has support in $\{x \in \mathbf{R}^n; |x| < 2\}$ and $\chi_n = 1$ for $|x| \leq 1$. For $\epsilon > 0$ small enough, we define

$$\begin{aligned} E_{L,\epsilon}(x) &= E_0(x) + \chi_n(\epsilon x)(E_L(x) - E_0(x)), \\ E_{S,\epsilon}(x) &= \chi_n(\epsilon x)E_S(x), \end{aligned}$$

and

$$V_\epsilon^r(y) = \chi_{|y|>r}(y)\chi_{n-1}(\epsilon y)(y \cdot \nabla_y \varphi(y) - \varphi(y)).$$

We further define an operator $B(\epsilon; \lambda)$ as

$$\begin{aligned} B(\epsilon; \lambda) &= -2\Delta + \lambda((x \cdot \nabla E_{L,\epsilon}) - nE_{S,\epsilon} + \nabla^* \cdot x E_{S,\epsilon} - E_{S,\epsilon} x \cdot \nabla \\ &\quad - T_\varphi^* V_\epsilon^r(y)(E_L^+ - E_L^-)T_\varphi - (R_\varphi^r)^*(E_L^+ - E_L^-)R_\varphi^r). \end{aligned}$$

We can consider that $B(\epsilon; \lambda)$ to be a bounded operator from $H^1(\mathbf{R}^n)$ to $H^{-1}(\mathbf{R}^n)$ (see Lemma 2.1). Moreover, for $u \in H^2(\mathbf{R}^n) \cap D(A)$ satisfying $Au \in H^1(\mathbf{R}^n)$, we define the commutator $[B(\epsilon; \lambda), A]$ as follows

$$\begin{aligned} &< i[B(\epsilon; \lambda), A]u, u > \\ &= \sum_{j=1}^4 < i[B_j(\epsilon; \lambda), A]u, u > \\ &= \sum_{j=1}^4 i(< Au, B_j(\epsilon; \lambda)u > - < B_j(\epsilon; \lambda)u, Au >), \end{aligned}$$

where

$$\begin{aligned} B_1(\epsilon; \lambda) &= -2\Delta + \lambda(x \cdot \nabla E_{L,\epsilon}), \\ B_2(\epsilon; \lambda) &= -\lambda((E_{S,\epsilon} x \cdot \nabla)^* + E_{S,\epsilon} x \cdot \nabla + nE_{S,\epsilon}) \\ &\quad = -\lambda(\nabla^* \cdot x E_{S,\epsilon} + E_{S,\epsilon} x \cdot \nabla + nE_{S,\epsilon}), \\ B_3(\epsilon; \lambda) &= -\lambda T_\varphi^* V_\epsilon^r(E_L^{+0} - E_L^{-0})T_\varphi, \\ B_4(\epsilon; \lambda) &= -\lambda(R_\varphi^r)^*(E_L^{+0} - E_L^{-0})R_\varphi^r. \end{aligned}$$

Then the $i[B(\epsilon; \lambda), A]$ has an extension $[B(\epsilon; \lambda), A]^0$ to be a bounded operator from $H^2(\mathbf{R}^n)$ to $H^{-2}(\mathbf{R}^n)$. Then we can prove the following lemma

LEMMA 2.5. Let $M(\epsilon; \lambda) = f_\delta(H(\lambda))B(\epsilon; \lambda)f_\delta(H(\lambda))$. Then $[M(\epsilon; \lambda), A]$ defined as a form on $D(A)$ is extended to a bounded operator on L^2 which is denoted by $[M(\epsilon; \lambda), A]^0$.

We can also prove the following lemma by a straightforward calculation.

LEMMA 2.6. Let $\lambda_0 - \delta < \lambda < \lambda_0 + \delta$, $0 < \epsilon < 1$. There exists a positive constant C independent of λ and ϵ such that

- (i) $\|(-\Delta + 1)^{-1/2}(B(\lambda) - B(\epsilon; \lambda))(-\Delta + 1)^{-1/2}\| \leq C\epsilon^\theta$,
- (ii) $\|(-\Delta + 1)^{-1/2}(d/d\epsilon)B(\epsilon; \lambda)(-\Delta + 1)^{-1/2}\| \leq C\epsilon^{\theta-1}$,
- (iii) $\|(-\Delta + 1)^{-1}[B(\epsilon; \lambda), A]^0(-\Delta + 1)^{-1}\| \leq C\epsilon^{\theta-1}$,

where $B(\lambda) = [H(\lambda), A]^0$.

SKETCH OF PROOF OF THEOREM 1.1(II) AND (III): By Theorem 1.1(i) and Lemma 2.3, we can take δ so small that

$$(2.11) \quad \begin{aligned} M(\lambda) &\equiv f_\delta(H(\lambda))i[H(\lambda), A]^0 f_\delta(H(\lambda)) \\ &\geq (\alpha/2)f_\delta(H(\lambda))^2 \end{aligned}$$

in the form sense.

Moreover, (2.11) together with Lemma 2.6(i), implies that

$$(2.12) \quad M(\epsilon; \lambda) \geq \gamma f_\delta(H(\lambda))^2$$

for $\epsilon > 0$ small enough, where $\gamma > 0$.

It follows from (2.12) that $M(\epsilon; \lambda)$ is non-negative and hence we define an operator, $G_\kappa(\epsilon; \lambda)$, on L^2 by

$$G_\kappa(\epsilon; \lambda) = (H(\lambda) - \lambda - i\kappa E(x) - i\epsilon M(\epsilon; \lambda))^{-1}$$

for $0 < \kappa < 1$ and $0 < \epsilon \ll 1$.

For $1/2 < \alpha \leq 1$, we write

$$F_\kappa(\epsilon; \lambda) = X_\alpha(\epsilon)G_\kappa(\epsilon; \lambda)X_\alpha(\epsilon),$$

where

$$X_\alpha(\epsilon) = (1 + |x|)^{-\alpha}(1 + \epsilon|x|)^{\alpha-1}.$$

Differentiating $F_\kappa(\epsilon; \lambda)$ in ϵ and using Lemma 2.3~Lemma 2.6, we have (see Tamura [13])

$$(2.13) \quad \|(d/d\epsilon)F_\kappa(\epsilon; \lambda)\| \leq C(\epsilon^{\alpha-1} + \epsilon^{\alpha-3/2}\|F_\kappa\|^{1/2} + \epsilon^{\theta-1}\|F_\kappa\|).$$

Let $\epsilon_0, 0 < \epsilon_0 \ll 1$. Then by (2.12), we have

$$(2.14) \quad \|F_\kappa(\epsilon_0; \lambda)\| \leq C\epsilon_0^{-1}.$$

By (2.13) and (2.14), we immediately obtain

$$(2.15) \quad \|F_\kappa(\epsilon; \lambda)\| \leq C,$$

where $C > 0$ independent of $0 < \kappa < 1$, $0 < \epsilon < \epsilon_0$ and $\lambda_0 - \delta < \lambda < \lambda_0 + \delta$. (2.13) and (2.15) imply Theorem 1.1(ii) and (iii). ■

3. The limiting amplitude principle.

We consider only the case $1 = a_-^{-2} < a_+^{-2}$. The other cases can be proved similarly. Theorem 1.2 is obtained as an immediate consequence of the following

LEMMA 3.1. *Let the pair (α, β) be as in Theorem 1.2. Then*

$$\|Q(\lambda, \pm i0; H(\lambda))^{-1}\|_{\beta \rightarrow -\alpha} = O(\lambda^{-d}), \quad (\lambda \rightarrow 0),$$

for some $d, 0 < d < 1/2$.

By the assumption (A.3), we can decompose $a^{-2}(x)$ as $a^{-2}(x) = E_1^\pm(x) + E_2(x)$ ($x \in \Omega_\pm$) in such a way :

$$(3.1) \quad \sum_{|\alpha| \leq 1} |x|^{|\alpha|} |\partial_x^\alpha (E_1^\pm(x) - a_\pm^{-2})| = O(|x|^{-\theta}) \quad (|x| \rightarrow \infty, x \in \Omega_\pm),$$

$$(3.2) \quad E_2(x) = O(|x|^{-1-\theta}) \quad (|x| \rightarrow \infty),$$

$$(3.3) \quad \sum_{|\alpha| \leq 1} \langle x \rangle^{|\alpha|} |\partial_x^\alpha (E_1^\pm(x) - a_\pm^{-2})| \leq \delta_0 \quad (x \in \Omega_\pm),$$

for $\delta_0 > 0$ small enough, δ_0 being fixed throughout.

LEMMA 3.2. *Let $H_1(\lambda) = -\Delta - \lambda(E_1(x) - 1)$, and $\alpha > 1$. Then, we have*

$$(3.4) \quad \|(H_1(\lambda) - \lambda \mp i\kappa a^{-2}(x))^{-1}\|_{\alpha \rightarrow -\alpha} = O(1), \quad (\lambda \rightarrow 0),$$

uniformly in $\kappa > 0$ small enough, where $E_1(x) = E_1^\pm(x)$ ($x \in \Omega_\pm$)

By interpolation, We can show that Lemma 3.1 follows from Lemma 3.2.

we also prove Lemma 3.2 by making use of commutator method. We define the commutator $i[H_1(\lambda), A]$ as a form on $H^2(\mathbf{R}^n) \cap D(A)$ as follows : For $u, v \in H^2(\mathbf{R}^n) \cap D(A)$,

$$\begin{aligned} & \langle i[H_1(\lambda), A]u, v \rangle \\ &= i(\langle Au, H_1(\lambda)v \rangle - \langle H_1(\lambda)u, Av \rangle). \end{aligned}$$

The key lemmas in this section are the following lemmas (Lemma 3.3 and 3.4).

LEMMA 3.3. *The form $i[H_1(\lambda), A]$ defined on $H^2(\mathbf{R}^n) \cap D(A)$ is extended to a bounded operator from $H^1(\mathbf{R}^n)$ to $H^{-1}(\mathbf{R}^n)$ which is denoted by $i[H_1(\lambda), A]^0$. Moreover we have*

$$\begin{aligned} & i[H_1(\lambda), A] \\ &= -2\Delta + \lambda(F_1 - T_\varphi^*(E_1^{+0} - E_1^{-0})\chi_{|y|>r}(y)(y \cdot \nabla_y \varphi(y) - \varphi(y))T_\varphi \\ & \quad - (R_\varphi^r)^*(E_1^{+0} - E_1^{-0})R_\varphi^r), \end{aligned}$$

where $0 < r \ll 1$ and $F_1 = x \cdot \nabla E_1(x) (x \in \Omega_{\pm})$.

LEMMA 3.4. Let $0 < \lambda \ll 1$, take $f_{\lambda}(p) \in C_0^{\infty}(\mathbf{R})$, $0 \leq f_{\lambda} \leq 1$ such that f_{λ} has support in $(\lambda/3, 3\lambda)$ and $f_{\lambda} = 1$ on $[\lambda/2, 2\lambda]$. Then, there exists a positive constant C which is independent of λ such that

$$(3.5) \quad f_{\lambda}(H_1(\lambda))i[H_1(\lambda), A]f_{\lambda}(H_1(\lambda)) \geq C\lambda f_{\lambda}(H_1(\lambda))^2.$$

in the form sense.

We can prove Lemma 3.3 in the same way as in the proof of Lemma 2.3. Here, we give only a sketch of the proof of Lemma 3.4. In order to show Lemma 3.4, we need the following lemma.

LEMMA 3.5. Let $u \in H^1(\mathbf{R}^n)$. Then, we have

$$|T_{\varphi}u|_{L^2(\mathbf{R}^{n-1})}^2 = \mp 2\text{Re} \int_{\Omega_{\pm}} u \overline{\partial_z} u dx.$$

PROOF: Let $u \in \mathbf{S}(\mathbf{R}^n)$. Noting that

$$\int_{\Omega_{\pm}} u \overline{\partial_z} u dx = \pm \int_{\mathbf{R}^{n-1}} \int_{\varphi(y)}^{\pm\infty} u \overline{\partial_z} u dz dy,$$

we have by integrating by parts

$$\int_{\Omega_{\pm}} u \overline{\partial_z} u dx = \mp \int_{\mathbf{R}^{n-1}} |T_{\varphi}u|^2 dy - \int_{\Omega_{\pm}} \partial_z u \overline{u} dx$$

Thus we have

$$\int_{\mathbf{R}^{n-1}} |T_{\varphi}u|^2 dy = \mp 2\text{Re} \int_{\Omega_{\pm}} u \overline{\partial_z} u dx$$

This implies lemma because $\mathbf{S}(\mathbf{R}^n)$ is dense in $H^1(\mathbf{R}^n)$. ■

SKETCH OF THE PROOF OF LEMMA 3.4: We simply write f_{λ} instead of $f_{\lambda}(H_1(\lambda))$. For $u \in L^2$, we have (for detail, see the proof of Lemma 2.3)

$$(3.6) \quad \begin{aligned} & \langle f_{\lambda}i[H_1(\lambda), A]f_{\lambda}u, u \rangle \\ &= 2 \langle \nabla f_{\lambda}u, \nabla f_{\lambda}u \rangle + \lambda \langle F_1 f_{\lambda}u, f_{\lambda}u \rangle \\ & \quad - \lambda \int_{\mathbf{R}^{n-1}} (y \cdot \nabla_y \varphi - \varphi)(E_1^{+0} - E_1^{-0}) |T_{\varphi}f_{\lambda}u|^2 dy. \end{aligned}$$

We estimate the second and the third term respectively. (3.3) implies that

$$(3.7) \quad |\langle F_1 f_{\lambda}u, f_{\lambda}u \rangle| \leq \delta_0 |f_{\lambda}u|_0^2.$$

Let $r > 0$ small enough. We decompose the third term as the following form

$$\begin{aligned} & \int_{\mathbf{R}^{n-1}} (y \cdot \nabla_y \varphi - \varphi)(E_1^{+0} - E_1^{-0}) |T_\varphi f_\lambda u|^2 dy \\ &= \int_{|y| < r} (y \cdot \nabla_y \varphi - \varphi)(E_1^{+0} - E_1^{-0}) |T_\varphi f_\lambda u|^2 dy \\ &+ \int_{|y| > r} (y \cdot \nabla_y \varphi - \varphi)(E_1^{+0} - E_1^{-0}) |T_\varphi f_\lambda u|^2 dy \end{aligned}$$

Repeating the argument in the proof of (2.10), we have

$$(3.8) \quad \left| \int_{|y| < r} (y \cdot \nabla \varphi - \varphi)(E_1^{+0} - E_1^{-0}) |T_\varphi f_\lambda u|^2 dy \right| \leq Cr^{\frac{n-1-\sigma p}{p}} (|\nabla f_\lambda u|_0^2 + |f_\lambda|_0^2),$$

for some $p, n-1 < p < \frac{n-1}{\sigma}$.

Noting that $\chi_{|y| > r}(y)(y \cdot \nabla \varphi - \varphi)$ is bounded, by Lemma 3.5, we have

$$\left| \int_{|y| > r} (y \cdot \nabla \varphi - \varphi)(E_1^{+0} - E_1^{-0}) |T_\varphi f_\lambda u|^2 dy \right| \leq C < |f_\lambda u|, |\partial_z f_\lambda u| >.$$

Moreover, let $\delta > 0$, small enough. Then we have

$$(3.9) \quad \left| \int_{|y| > r} (y \cdot \nabla \varphi - \varphi)(E_1^{+0} - E_1^{-0}) |T_\varphi f_\lambda u|^2 dy \right| \leq \delta |f_\lambda u|_0^2 + 1/\delta |\nabla f_\lambda u|_0^2.$$

Noting $0 < \lambda, r, \delta, \delta_0 \ll 1$, by (3.6) \sim (3.9), we have (3.5). ■

We define

$$E_{1,\epsilon}(x) = E_0(x) + \chi_n(\epsilon x)(E_1(x) - E_0(x)),$$

where $\epsilon > 0$ small enough. We further define an operator $C(\epsilon; \lambda)$ as

$$\begin{aligned} C(\epsilon; \lambda) &= -2\Delta + \lambda(F_{1,\epsilon} - T_\varphi^*(E_1^{+0} - E_1^{-0})V_\epsilon^r(y)T_\varphi \\ &\quad - (R_\varphi^r)^*(E_1^{+0} - E_1^{-0})R_\varphi^r), \end{aligned}$$

where $F_{1,\epsilon} = x \cdot \nabla E_{1,\epsilon}(x)$ ($x \in \Omega_\pm$). We can consider $C(\epsilon; \lambda)$ to be a bounded operator from $H^1(\mathbf{R}^n)$ to $H^{-1}(\mathbf{R}^n)$. $u \in H^2(\mathbf{R}^n) \cap D(A)$ satisfying $Au \in H^1(\mathbf{R}^n)$, we define the commutator $i[C(\epsilon; \lambda), A]$ as follows

$$\begin{aligned} & \langle i[C(\epsilon; \lambda), A]u, u \rangle \\ &= \sum_{j=1}^3 \langle i[C_j(\epsilon; \lambda), A]u, u \rangle \\ &= \sum_{j=1}^3 (\langle Au, C_j(\epsilon; \lambda)u \rangle - \langle C_j(\epsilon; \lambda)u, Au \rangle), \end{aligned}$$

where

$$\begin{aligned} C_1(\epsilon; \lambda) &= -2\Delta + \lambda F_{1,\epsilon}, \\ C_2(\epsilon; \lambda) &= -\lambda T_\varphi^*(E_1^{+0} - E_1^{-0})V_\epsilon^r T_\varphi, \\ C_3(\epsilon; \lambda) &= -\lambda(R_\varphi^r)^*(E_1^{+0} - E_1^{-0})R_\varphi^r. \end{aligned}$$

We can also show that $i[C(\epsilon; \lambda), A]$ is extended to a bounded operator $H^2(\mathbf{R}^n)$ to $H^{-2}(\mathbf{R}^n)$. Moreover, we have the following (see Lemma 2.5)

LEMMA 3.6. *Let $N(\epsilon; \lambda) = f_\lambda(H(\lambda))C(\epsilon; \lambda)f_\lambda(H(\lambda))$. Then $[N(\epsilon; \lambda), A]$ defined as a form on $D(A)$ is extended to a bounded operator on L^2 which is denoted by $[N(\epsilon; \lambda), A]^0$.*

We can prove the following lemma by a straightforward calculation.

LEMMA 3.7. *As $\lambda \rightarrow 0$, one has*

- (i) $\|(-\Delta + \lambda)^{-1/2}(C(\epsilon; \lambda) - C(\lambda))(-\Delta + \lambda)^{-1/2}\| = \epsilon^\theta O(1)$,
 - (ii) $\|(-\Delta + \lambda)^{-1/2}(\frac{d}{d\epsilon}C(\epsilon; \lambda))(-\Delta + \lambda)^{-1/2}\| = \epsilon^{\theta-1}O(1)$,
 - (iii) $\|(-\Delta + \lambda)^{-1}[C(\epsilon; \lambda), A]^0(-\Delta + \lambda)^{-1}\| = \epsilon^{\theta-1}O(\lambda^{-1}) + O(\lambda^{-1})$,
- where $C(\lambda) = i[H_1(\lambda), A]^0$.

SKETCH OF PROOF OF LEMMA 3.2: Lemma 3.4 and 3.7 (i) imply that

$$(3.10) \quad N(\epsilon; \lambda) \geq \gamma \lambda f_\lambda^2$$

for $\epsilon > 0$ small enough, where $\gamma > 0$ is independent of ϵ, λ . It follows from (3.10) that $C(\epsilon; \lambda)$ is non-negative and hence we define an operator, $J_\kappa(\epsilon; \lambda)$, on L^2 by

$$J_\kappa(\epsilon; \lambda) = (H_1(\lambda) - \lambda - i\kappa a^{-2}(x) - i\epsilon N(\epsilon; \lambda))^{-1}$$

for $\kappa, 0 < \kappa < 1$ and $\epsilon > 0$ small enough.

We write

$$E_\kappa(\epsilon; \lambda) = X_1 J_\kappa(\epsilon; \lambda) X_1,$$

where $X_1 = (1 + |x|^2)^{-1/2}$.

Differentiating $E_\kappa(\epsilon; \lambda)$ in ϵ and using Lemma 3.3~ 3.7, we have

$$(3.11) \quad \|(d/d\epsilon)E_\kappa(\epsilon; \lambda)\| \leq C(1 + \epsilon^{-1/2}\|E_\kappa\|^{1/2} + \epsilon^{\theta-1}\|E_\kappa\|).$$

Let $\epsilon_0, 0 < \epsilon_0 \ll 1$. and

$$(3.12) \quad \|E_\kappa(\epsilon_0; \lambda)\| = \epsilon_0^{-1}O(1), \quad (\lambda \rightarrow 0).$$

(3.11) and (3.12) imply

$$\|E_\kappa(0; \lambda)\| = O(1), \quad (\lambda \rightarrow 0),$$

uniformly $\kappa, 0 < \kappa < 1$. Thus, we obtain Lemma 3.2. ■

Remark. In order to show (3.12), we need the well-known inequality

$$(3.13) \quad \int_{\mathbf{R}^n} \langle x \rangle^{-2} |u(x)|^2 dx \leq C \int_{\mathbf{R}^n} |\nabla u(x)|^2 dx,$$

where $n \geq 3$.

REFERENCES

1. Ben-Artzi. M, Dermenjian. Y and Guillot. J.C, *Acoustic waves in perturbed stratified fluids: a spectral theory*, Comm. Partial Differential Equations **14** (1989), 479–517.
2. Debièvre. S and Pravica. W, *Spectral analysis for optical fibres and stratified fluids I : the limiting absorption principle*, J. Func. Anal **98** (1991), 404–436.
3. Dermenjian. Y and Guillot. J.C, *Théorie spectrale de la propagation des ondes acoustiques dans un milieu stratifié perturbé*, J. Defferential Equations **62** (1986), 357–409.
4. Eastham. M.S and Kalf. H, "Schrödinger-type operators with continuous spectra, Resaech note in Math," Pitman, 1982.
5. Eidus. D, *The principle of limiting amplitude*, Russ. Math. Surv **24** (1969), 97–167.
6. ———, *The limiting absorption and amplitude principle for the diffraction problem with two unbounded media*, Comm. Math. Phys **107** (1986), 27–38.
7. Froese. R and Herbst. I, *Exponential bounds and absence of positive eigenvalues for N-body Schrödinger operator*, Comm. Math. Phys **87** (1982), 429–447.
8. Iwashita. H, *Spectral theory for symmetric systems in an exterior domain*, Tsukuba J. Math **11** (1987), 241–256.
9. Kikuchi. K and Tamura. H, *Limiting amplitude principle for acoustic propagators in perturbed stratified fluids*, J. Defferential Equations **93** (1991), 260–282.
10. Mourre. E, *Absence of singular continuous spectrum for certain self-adjoint operators*, Comm. Math. Phys **78** (1981), 391–408.
11. Perry. P, Sigal. I.M and Simon. B, *Spectral analysis of N-body Schrödinger operators*, Ann of Math **114** (1981), 519–567.
12. Tamura. H, *Resolvent estimates at low frequencies and limiting amplitude principle for acoustic propagator*, J. Math. Soc. Japan **41** (1989), 549–575.
13. ———, *Principle of limiting absorption for N-body Schrödinger operators, A remark on the commutator method*, Lett. in Math. Phys **17** (1989), 31–36.
14. Weder. R, *Spectral analysis of strongly propagator systems*, J. Reine Angew. Math. **354** (1984), 95–122.
15. ———, *Spectral and scattering theory in perturbed stratified fluids*, J. Math. Pures et Appl **64** (1985), 149–173.
16. ———, *Spectral and scattering theory in perturbed stratified fluids II, transmission and exterior domains*, J. Defferential Equations **64** (1986), 109–131.
17. ———, *The limiting absorption principle at thresholds*, J. Math. pures et Appl **67** (1988), 318–338.
18. Wilcox. C. H, *Spectral analysis of the Pekeris operator in the theory of acoustic wave propagation in Shallow water*, Arch. Rational Mech. Anal **60** (1976), 259–300.